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A discretization of the matrix nonlinear Schrödinger equation

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Abstract. In this paper, we shall show that a class of solutions to the discrete coupled matrix nonlinear Schrödinger equation (DCMNLSE) is gauge equivalent to the discrete equation of the Schrödinger flow of maps into the Grassmannian and the realizing gauge transformation is only the discretization of a classical gauge transformation between the matrix nonlinear Schrödinger equation (MNLSE) and the Schrödinger flow of maps into the Grassmannian. In other words, from the viewpoint of gauge equivalence, a class of solutions of the DCMNLSE is a correct discretization of the MNLSE.

1. Introduction

The nonlinear Schrödinger equation (NLSE), $i\psi_t + \psi_{xx} + 2\kappa|\psi|^2\psi = 0$, where the subscripts denote partial derivatives and κ is a real constant, arises in physics from a variety of backgrounds, such as in plasma physics and nonlinear optics, and provides a fairly universal model of a nonlinear equation. The following interesting generalization of the NLSE:

$$iq_t + q_{xx} + 2qq^*q = 0 \qquad (1)$$

was first studied by Fordy and Kulish in [1], where q is a map from R^2 to the space $M_{(m-k)\times k}$ of $(m-k)\times k$ complex matrices, $1 \leqslant k \leqslant m-1$ and q^* denotes the complex transposed conjugate matrix of q. We will follow [2] by calling this equation, in this paper, the matrix nonlinear Schrödinger equation (MNLSE) when $k \geqslant 2$ or $m-k \geqslant 2$. Note that if q is a 1×1 complex matrix, (1) is just the NLSE equation with $\kappa = 1$ ($NLSE^+$). The MNLSE is also applied in many fields. For example, when q is a 1×2 complex matrix, the corresponding equation (1) is called the 2-vector or 2-component NLSE, which is very useful in nonlinear fibre communications (see [3]). A systematic study of the 2-vector NLSE can be found in [4–6].

On the other hand, the study of nonlinear integrable differential-difference equations has received considerable attention in recent years (see, e.g., [7, 8]). The integrable discrete nonlinear Schrödinger equation (DNLSE) $i(dq_n/dt)+(q_{n+1}+q_{n-1}-2q_n)+\kappa|q_n|^2(q_{n+1}+q_{n-1}) = 0$ was introduced by Ablowitz and Ladik [9] who constructed the discrete version of the AKNS system. The DNLSE also has a rather wide area of application; see, e.g., [10] for a listing of its physical applications. The parallel generalization of the $DNLSE^+$ to that of the $NLSE^+$ is naturally introduced as follows:

$$i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}q_n^*q_n + q_nq_n^*q_{n-1}) = 0 \qquad (2)$$

which is called the discrete-matrix nonlinear Schrödinger equation (DMNLSE).

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The concept of gauge equivalence between completely integrable equations was introduced in [11, 12] and then became an important tool in the study of solitons [13]. From a recent work due to the author [14], we know that the NLSE for $\kappa = 1, 0$ and -1 is exactly gauge equivalent to the Schrödinger flow of maps into the Euclidean 2-space $S^2 \hookrightarrow R^3$ (with Gauss curvature 1) (i.e. the HF model) [12], the complex plane C (with Gauss curvature 0) and the hyperbolic 2-space $H^2 \hookrightarrow R^{2+1}$ (with Gauss curvature -1) (the M-HF model) [14], respectively. This gives a beautiful unified geometric explanation for the NLSE. Analogous results for the (2+1)-dimensional case can be found in [15, 16]. The MNLSE (1) is now shown to be gauge equivalent to the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ from the recent results due to Langer and Perline [17] and Terng and Uhlenbeck [2]. The corresponding results for $G_{1,2} = CP^1 = S^2$ is exactly the case described in [12] or [14]. However, generally speaking, in classical integrable theory we have a number of remarkable properties which may not exist in their discrete counterparts. But, in 1982, Ishimori showed in [18] that the DNLSE$^+$ is gauge equivalent to the discrete HF model (DHF), which reads

$$\mathrm{d}\boldsymbol{S}_n/\mathrm{d}t = -\frac{2\boldsymbol{S}_{n+1} \times \boldsymbol{S}_n}{1 + \boldsymbol{S}_{n+1} \cdot \boldsymbol{S}_n} + \frac{2\boldsymbol{S}_n \times \boldsymbol{S}_{n-1}}{1 + \boldsymbol{S}_n \cdot \boldsymbol{S}_{n-1}} \tag{3}$$

where $\boldsymbol{S}_n = (s_n^1, s_n^2, s_n^3) \in R^3$ with $|\boldsymbol{S}_n|^2 = (s_n^1)^2 + (s_n^2)^2 + (s_n^3)^2 = 1$, with $\cdot$ and $\times$ denoting the inner and the cross product in R^3. Furthermore, by finding the new Lax pairs, the author proved in [19] that the DNLSE$^-$, i.e. the DNLSE with $\kappa = -1$, (resp. DNLSE$^+$) is gauge equivalent to the DM-HF (resp. DHF) and, meanwhile, the continuous limit of the realizing gauge transformations is just a classical limit between the NLSE$^-$ (resp. NLSE$^+$) and the M-HF model (resp. HF model). This reveals that there is a reconciliation of the gauge equivalent structures of the DNLSE and the NLSE for $\kappa = 1$ and -1. It should be pointed out that the Ishimori's gauge transformation in [18] is not the discretization of a classical gauge transformation between the NLSE$^+$ and the HF model (see [19]). After a thorough understanding of the gauge equivalent structures of the NLSE, the DNLSE (which is the discrete gauge equivalent corresponding to the NLSE) and the MNLSE, one would like naturally to see the discrete counterpart of the gauge equivalent structure of the MNLSE according to the correspondence principle in quantum dynamics. Originally formulated by Bohr, the correspondence principle, which states that a new (physical) theory must explain all phenomena that the older, superseded theory could explain, was initially used to describe the relationship between quantum theory and classical physics. During the early days of quantum theory, physicists used the correspondence principle to formulate their theories so that in situations where classical physics is valid, their theories describing quantum phenomena reduced to the same equations obtained from classical physics. The correspondence principle is valid for many areas of quantum theory, and also applies to other theories.

In this paper, we shall show that a class of solutions to the following (integrable) discrete coupled matrix nonlinear Schrödinger equation (DCMNLSE):

$$\begin{cases} \mathrm{i}(\mathrm{d}q_n/\mathrm{d}t) + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}r_nq_n + q_nr_nq_{n-1}) = 0 \\ -\mathrm{i}(\mathrm{d}r_n/\mathrm{d}t) + (r_{n+1} + r_{n-1} - 2r_n) + (r_{n+1}q_nr_n + r_nq_nr_{n-1}) = 0 \end{cases} \tag{4}$$

which is the discretization of the (integrable) coupled matrix nonlinear Schrödinger equation (CMNLSE)

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2qrq = 0 \\ -\mathrm{i}r_t + r_{xx} + 2rqr = 0 \end{cases} \tag{5}$$

is gauge equivalent to the discrete equation of the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (see equation (11) in the next section), and the continuous limit of the realizing gauge transformation is exactly a classical gauge transformation between the MNLSE (1) and the Schrödinger flow of maps into the Grassmannian $G_{k,m}$, where q_n is a map from R^2 to the space $M_{(m-k)\times k}$ of $(m-k)\times k$ complex matrices and r_n is a map to the space $M_{k\times(m-k)}$ of $k\times(m-k)$ complex matrices. When $r_n = q_n^*$, equation (4) reduces to equation (2). This reflects that, from the viewpoint of gauge equivalence, a class of solutions to the DCMNLSE (4) is a discretization of the MNLSE (1) such that it corresponds to the discrete gauge equivalent counterpart of the MNLSE.

This paper is organized as follows. In section 2 we present the desired Lax pairs for the DCMNLSE and the discrete equation of the Schrödinger flow of maps into the Grassmannian. In section 3 we show the main result of this paper and in section 4 we give an example to illustrate the result.

2. Lax pairs and their continuous limits

Similar to the Lax pairs for the DNLSE⁺ and DHF in [19], in this section we shall present Lax pairs for the DCMNLSE and the discrete equation of the Schrödinger flow of maps into the Grassmannian such that they are exactly the discretizations of their corresponding classical Lax pairs.

In order to solve the DCMNLSE (4), we usually need to add the zero-boundary conditions $q_n \to 0$ and $r_n \to 0$ as $n \to \infty$. Following the conventional terminology (see [13]), we also set the zero-boundary condition to be a rapidly decreasing boundary condition. Equation (4) allows a Lax pair as follows:

$$\phi_{n+1} = L_n\phi_n \qquad \mathrm{d}\phi_n/\mathrm{d}t = M_n\phi_n \tag{6}$$

with

$$L_n = \begin{pmatrix} zI_k & r_n z^{-1} \\ -q_n z & z^{-1}I_{m-k} \end{pmatrix}$$

$$M_n = \mathrm{i}\begin{pmatrix} (1-z^2+z-z^{-1})I_k - r_n q_{n-1} & -r_n + r_{n-1}z^{-2} \\ -q_n + q_{n-1}z^2 & (-1+z^{-2}+z-z^{-1})I_{m-k} + q_n r_{n-1} \end{pmatrix}$$

where z is a spectral parameter. In fact, one may verify that the compatibility condition

$$\mathrm{d}L_n/\mathrm{d}t + L_nM_n - M_{n+1}L_n = 0 \tag{7}$$

of (6) yields only (4).

As usual (see, e.g., [9, 19]), the continuous limit ($\Delta x \to 0$; Δx being the discretization parameter) of the Lax pair (6) is

$$\phi_x = L\phi \qquad \phi_t = M\phi \tag{8}$$

with

$$L = \lambda\sigma_3 + U \qquad M = -\mathrm{i}2\lambda^2\sigma_3 - 2\mathrm{i}\lambda U + \mathrm{i}(U^2 + U_x)\sigma_3 \tag{9}$$

and $U = \begin{pmatrix} 0 & r \\ -q & 0 \end{pmatrix}$, after the substitution

$$z \to 1 + \lambda\Delta x \qquad q_n \to q\Delta x \qquad r_n \to r\Delta x \qquad n\Delta x = x(\text{fixed}) \qquad t\Delta x^2 \to t \tag{10}$$

(λ is a parameter) and setting $\phi_n \sim \phi$, expanding $q_{n\pm1} \sim \Delta x(q \pm \Delta x q_x + \frac{\Delta x^2}{2}q_{xx} \pm \cdots)$, etc. It can be directly verified that the integrability condition of (8) yields simply the CMNLSE (5) and if $r = q^*$, then (5) reduces to the MNLSE (1). One may refer to [20] for a study of equation (5) in the case of $k = 1, m = 2$ and its physical applications.

The following differential-difference equation:

$$dS_n/dt = 4i(I + S_n S_{n-1})^{-1} - 4i(I + S_{n+1} S_n)^{-1} \quad (11)$$

is the discrete equation of the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (15) (see below), where $I = I_m$ denotes the $m \times m$ unit matrix and S_n is of the form

$$U_n^{-1} \sigma_3 U_n \quad (12)$$

with U_n being an $m \times m$ unitary matrix and $\sigma_3 = \begin{pmatrix} I_k & 0 \\ 0 & -I_{m-k} \end{pmatrix}$. When $k = 1$ and $m = 2$, i.e. $G_{k,m} = CP^1 = S^2$, equation (11) reduces to the DHF (3). We usually add the boundary condition $S_n \to \sigma_3$ as $n \to \infty$ in solving this equation. Now we shall put aside the cumbersome but straightforward calculations and present the final results. Equation (11) permits the following Lax pair:

$$\psi_{n+1} = \tilde{L}_n \psi_n \qquad d\psi_n/dt = \tilde{M}_n \psi_n \quad (13)$$

with $\tilde{L}_n = \frac{z+z^{-1}}{2} I + \frac{z-z^{-1}}{2} S_n$ and $\tilde{M}_n = i2(1 - \frac{z^2+z^{-2}}{2})(I + S_n S_{n-1})^{-1} S_n + i(z - z^{-1})I - i(z^2 - z^{-2})(I + S_n S_{n-1})^{-1}$. After the substitution $z \to 1 + \lambda\Delta x$, $\Delta x \to 0$, $t\Delta x^2 \to t$, $n\Delta x = x$ (fixed), $S_n \to S$ and $\psi_n \to \psi$, the continuous limit of (13) is

$$\psi_x = \tilde{L}\psi \qquad \psi_t = \tilde{M}\psi \quad (14)$$

with $\tilde{L} = \lambda S$, $\tilde{M} = -i2\lambda^2 S + i\lambda S_x S$ and S satisfying $S^2 = I$. The integrability condition of (14) reads

$$S_t = \frac{1}{2i}[S, S_{xx}] \quad (15)$$

which is equivalent to the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ displayed in [2]

$$\gamma_t = [\gamma, \gamma_{xx}] \qquad \gamma \in G_{k,m} \quad (16)$$

where $G_{k,m}$ is regarded as the adjoint $U(m)$-orbit at $a = \begin{pmatrix} \frac{i}{2} I_k & 0 \\ 0 & -\frac{i}{2} I_{m-k} \end{pmatrix}$ in the Lie algebra $u(m)$ of $U(m)$, i.e. $G_{k,m} = \mathrm{Ad}(U(m))a = \{U^{-1}aU | U \in U(m)\}$.

3. Gauge equivalence

In this section, by using the Lax pairs displayed in the preceding section, we shall show that there is a gauge transformation between (a class of solutions to) the DCMNLSE (4) and the discrete equation (11) of the Schrödinger flow of maps into $G_{k,m}$, and the continuous limit of the realizing gauge transformation is just a classical limit between the MNLSE (1) and the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (15).

First we suppose $\{S_n\}$ is of the form (12) and fulfils the discrete equation (11) of the Schrödinger flow of maps into $G_{k,m}$. We then choose a sequence of $m \times m$-matrices $\{G_n(t)\}$ such that $\sigma_3 = G_n S_n G_n^{-1}$ and $\forall n$

$$G_{n+1} G_n^{-1} = \begin{pmatrix} I_k & r_n(t) \\ -q_n(t) & I_{m-k} \end{pmatrix} \quad (17)$$

for some $(m - k) \times k$ complex matrix $q_n(t)$ and $k \times (m - k)$ complex matrix $r_n(t)$. In fact, because of (12), there exists a sequence of unitary matrices $\{U_n\}$ such that $S_n = U_n^* \sigma_3 U_n$. It is obvious that the general solutions to $\sigma_3 = G_n S_n G_n^{-1}$ are of the form

$$G_n = \mathrm{diag}(A_n, B_n) U_n \quad (18)$$

where A_n is a non-singular $k \times k$ matrix and $\{B_n\}$ is a non-singular $(m-k) \times (m-k)$ matrix. Now we first fix A_0 and B_0, which will be restricted in remark 2 below, and then come to prove that, $\forall n \neq 0$, A_n and B_n can be chosen progressively such that (17) holds for some q_n and r_n. Substituting (18) into (17), we obtain

$$\begin{pmatrix} A_{n+1}^{-1} & 0 \\ 0 & B_{n+1}^{-1} \end{pmatrix} \begin{pmatrix} I_k & r_n \\ -q_n & I_{m-k} \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} = U_{n+1} U_n^*. \tag{19}$$

If we denote U_n by $\begin{pmatrix} U_n^1 & U_n^2 \\ U_n^3 & U_n^4 \end{pmatrix}$, then we see that (19) can be rewritten as

$$A_{n+1}^{-1} A_n = U_{n+1}^1 U_n^{1*} + U_{n+1}^2 U_n^{2*} \tag{20}$$

$$B_{n+1}^{-1} B_n = U_{n+1}^3 U_n^{3*} + U_{n+1}^4 U_n^{4*} \tag{21}$$

$$r_n = A_n p_n^* B_n^{-1} \tag{22}$$

$$q_n = B_n p_n A_n^{-1} \tag{23}$$

$$p_n = (U_{n+1}^3 U_n^{3*} + U_{n+1}^4 U_n^{4*})^{-1} (U_{n+1}^3 U_n^{1*} + U_{n+1}^4 U_n^{2*}) \tag{24}$$

where the inversibility of $U_{n+1}^3 U_n^{3*} + U_{n+1}^4 U_n^{4*}$ is due to the fact that $I + S_{n+1} S_n$ $(= U_{n+1}^* (U_{n+1} U_n^* + \sigma_3 U_{n+1} U_n^* \sigma_3) U_n)$ is inversible in equation (11). Hence, we may choose A_n and B_n for $n \neq 0$ progessively by relations (20) and (21) and choose r_n and q_n by (22) and (23). This proves the existence of G_n. Now, we put

$$\begin{aligned} L_n^G(z) &= G_{n+1} \tilde{L}_n(z) G_n^{-1} = \begin{pmatrix} z I_k & r_n z^{-1} \\ -q_n z & z^{-1} I_{m-k} \end{pmatrix} \\ M_n^G(z) &= \mathrm{d}G_n/\mathrm{d}t\, G_n^{-1} + G_n \tilde{M}_n(z) G_n^{-1} \\ &= \mathrm{d}G_n/\mathrm{d}t\, G_n^{-1} + \mathrm{i} \begin{pmatrix} (1 - z^2 + z - z^{-1}) I_k & r_{n-1}(z^{-2} - 1) \\ q_{n-1}(z^2 - 1) & (-1 + z^{-2} + z - z^{-1}) I_{m-k} \end{pmatrix} \end{aligned}$$

where $\tilde{L}_n(z)$ and $\tilde{M}_n(z)$ are the coefficients in the Lax pair (13). Since $\tilde{L}_n$ and $\tilde{M}_n$ satisfy the integrability condition of (13), we have

$$\frac{\mathrm{d}L_n^G}{\mathrm{d}t} + L_n^G M_n^G - M_{n+1}^G L_n^G = 0. \tag{25}$$

If we let $\frac{\mathrm{d}G_n}{\mathrm{d}t} G_n^{-1} = \mathrm{i} \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \pi_n \end{pmatrix}$, where α_n, β_n, γ_n and π_n are temporally arbitrary, then the vanishing of the diagonal part in (25) leads to

$$\begin{aligned} &\beta_n = -r_n + r_{n-1} \qquad \gamma_n = -q_n + q_{n-1} \\ &\alpha_n + r_n q_{n-1} = \alpha_{n+1} + r_{n+1} q_n \qquad \pi_n - q_n r_{n-1} = \pi_{n+1} - q_{n+1} r_n \qquad \forall n. \end{aligned}$$

In other words, from the above relations we have

$$\frac{\mathrm{d}G_n}{\mathrm{d}t} G_n^{-1} = \mathrm{i} \begin{pmatrix} -r_n q_{n-1} & -r_n + r_{n-1} \\ -q_n + q_{n-1} & q_n r_{n-1} \end{pmatrix} + \mathrm{i} \begin{pmatrix} \tau(t) & 0 \\ 0 & \sigma(t) \end{pmatrix} \tag{26}$$

for some $k \times k$ matrix $\tau(t)$ and $(m-k) \times (m-k)$ matrix $\sigma(t)$ which do not depend on n, but may depend on A_0 and B_0. Note that the above restrictions on G_n allow an arbitrariness in G_n of the form

$$G_n \to \begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix} G_n \tag{27}$$

for some non-singular matrices $P(t)$ and $Q(t)$ depending only on t. In fact, denoting $\tilde{G}_n = \begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix} G_n$ under this transformation, we then have $\tilde{G}_{n+1} \tilde{G}_n^{-1} = \begin{pmatrix} 1 & \tilde{r}_n \\ -\tilde{q}_n & 1 \end{pmatrix}$

with $\tilde{q}_n = Q(t)q_n P(t)^{-1}$ and $\tilde{r}_n = P(t)r_n Q(t)^{-1}$. Meanwhile, a straightforward calculation shows

$$\frac{\mathrm{d}\tilde{G}_n}{\mathrm{d}t}\tilde{G}_n^{-1} = \mathrm{i}\begin{pmatrix} -\tilde{r}_n\tilde{q}_{n-1} & -\tilde{r}_n + \tilde{r}_{n-1} \\ -\tilde{q}_n + \tilde{q}_{n-1} & \tilde{q}_n\tilde{r}_{n-1} \end{pmatrix} + \begin{pmatrix} P_t P^{-1} + P\mathrm{i}\tau P^{-1} & 0 \\ 0 & Q_t Q^{-1} + Q\mathrm{i}\sigma Q^{-1} \end{pmatrix}.$$

If we require $P(t)$ and $Q(t)$ to satisfy

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = -\mathrm{i}P(t)\tau(t) \qquad \frac{\mathrm{d}Q}{\mathrm{d}t}(t) = -\mathrm{i}Q(t)\sigma(t)$$

then G_n can be modified so that for the new G_n the second term on the right of (26) vanishes. This implies that $M_n^G(z)$ is exactly the second coefficient in (6) and $\{(q_n, r_n)\}$ satisfies the DCMNLSE (4).

Remark 1. All the couples of sequences $\{(q_n, r_n)\}$, being given by (22) and (23) for some sequences of $(m-k)\times k$-matrices $\{p_n\}$, unitary $m\times m$-matrices $\{U_n\}$, non-singular $k\times k$-matrices $\{A_n\}$ and non-singular $(m-k)\times(m-k)$-matrices $\{B_n\}$ with relations (20), (21) and (24), such that

$$G_n = \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} U_n \tag{28}$$

solves

$$\begin{aligned} G_{n+1} &= \begin{pmatrix} I_k & r_n \\ -q_n & I_{m-k} \end{pmatrix} G_n \\ \frac{\mathrm{d}G_n}{\mathrm{d}t} &= \mathrm{i}\begin{pmatrix} -r_n q_{n-1} & -r_n + r_{n-1} \\ -q_n + q_{n-1} & q_n r_{n-1} \end{pmatrix} G_n \end{aligned} \tag{29}$$

consist of a class of solutions to the DCMNLSE (4). When $k = 1$ and $m = 2$ (i.e. in the case of $G_{1,2} = S^2$), as displayed in [19], we get $A_n = \bar{B}_n$ and therefore $r_n = q_n^*$ from (22) and (23). But when $k \geqslant 2$ or $m-k \geqslant 2$, as will be illustrated by an example in the next section, it is impossible to get $r_n = q_n^*$ in general. So the DCMNLSE appears very naturally when exploring the gauge equivalent structure of the discrete equation of the Schrödinger flow of maps into the Grassmannian in this case.

Remark 2. At the end of this section, we shall show that the continuous limits of A_n and B_n are unitary matrices, which is equivalent to saying that the continuous limits of A_0 and B_0 (i.e. $\lim_{\Delta x\to 0} A_0$ and $\lim_{\Delta x\to 0} B_0$) are unitary matrices. Therefore we will require the sequences $\{A_n\}$ and $\{B_n\}$ with the restriction that the continuous limits of A_0 and B_0 are unitary matrices and hence $r = q^*$ after taking the continuous limit from (22) and (23). If $\{(q_n, r_n)\}$ is a couple of the sequences given in remark 1 with the above restriction, then, after taking the continuous limit in both sides of (26), we see that the two t-depending functions $\mathrm{i}\tau$ and $\mathrm{i}\sigma$ have their continuous limits τ_0 and σ_0 (i.e. $\mathrm{i}\tau \to \Delta x^2\tau_0$ and $\mathrm{i}\sigma \to \Delta x^2\sigma_0$) in $u(k)$ and $u(m-k)$, respectively, where $u(k)$ is the Lie algebra of the unitary group of degree k, etc. Thus the continuous limits of the corresponding P and Q which appeared in (27) are automatically unitary matrices by the equations they satisfy. This indicates that, under these circumstances, the relation $r = q^*$ is still preserved by transformation (27) for $\{(q_n, r_n)\}$. Hence the above restriction on A_0 and B_0 is very natural.

Next, we prove that the above process from the discrete equation (11) of the Schrödinger flow of maps into $G_{k,m}$ to the class of solutions (see remark 1) of the DCMNLSE (4) is reversible. Suppose $\{(q_n(t), r_n(t))\}$, being given in remark 1, is a solution to the DCMNLSE equation (4). The corresponding solution to Lax pair (6) is denoted by $\{\phi_n(t, z)\}$. It is easy to see from (29) that $\{G_n\}$ is in fact a fundamental solution to (6) at $z = 1$.

Now, we consider the following gauge transformation:

$$\phi_n(t,z) = G_n(t)\psi_n(t,z) \tag{30}$$

and come to prove that the above $\{\psi_n(t,z)\}$ is a solution to Lax pair (13) of (11). In order to do this, we put $\psi_{n+1} = \tilde{L}_n\psi_n$ and $\mathrm{d}\psi_n/\mathrm{d}t = \tilde{M}_n\psi_n$ for some $\tilde{L}_n$ and $\tilde{M}_n$. Applying the first equation of Lax pair (6), from (30) we have

$$\tilde{L}_n = G_{n+1}^{-1}L_nG_n. \tag{31}$$

Then substituting $G_{n+1} = \begin{pmatrix} I_k & r_n \\ -q_n & I_{m-k} \end{pmatrix} G_n$ into (31), we obtain

$$\tilde{L}_n = \frac{z+z^{-1}}{2}I + \frac{z-z^{-1}}{2}G_n^{-1}\sigma_3G_n := \frac{z+z^{-1}}{2}I + \mathrm{i}\frac{z-z^{-1}}{2}S_n$$

where $S_n = G_n^{-1}\sigma_3G_n$ with $S_n^2 = I$. In what follows, we have to show that $S_n \in G_{k,m}$, or in other words, $\forall n$, $S_n = U_n^*\sigma_3U_n$ for some unitary matrix U_n. In fact, $\forall n$ we set

$$U_n = \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix}^{-1} G_n$$

which is a unitary matrix from (28) in remark 1, and it is straightforward to obtain the fact that $S_n = G_n^{-1}\sigma_3G_n = U_n^*\sigma_3U_n \in G_{k,m}$.

Using the second Lax equation for ϕ_n, we have

$$\begin{aligned}\tilde{M}_n &= G_n^{-1}M_nG_n - G^{-1}\mathrm{d}G_n/\mathrm{d}t = G_n^{-1}(t)(M_n(t,z) - M_n(t,1))G_n(t) \\ &= \mathrm{i}\left(1 - \frac{z^2+z^{-2}}{2}\right)G_{n-1}^{-1}\sigma_3G_n + \mathrm{i}(z-z^{-1})I - \mathrm{i}\frac{z^2-z^{-2}}{2}G_{n-1}^{-1}G_n \\ &= \mathrm{i}2\left(1 - \frac{z^2+z^{-2}}{2}\right)(I+S_nS_{n-1})^{-1}S_n + \mathrm{i}(z-z^{-1})I - \mathrm{i}(z^2-z^{-2})(I+S_nS_{n-1})^{-1}\end{aligned}$$

where we have used the identity $G_{n-1}^{-1}G_n = 2(I + S_nS_{n-1})^{-1}$. From this we see that the above $\tilde{L}_n$ and $\tilde{M}_n$ are exactly the same coefficients as in (13) for S_n being given by $S_n = G_n^{-1}\sigma_3G_n = U_n^*\sigma_3U_n$. Hence $\{\psi_n\}$ is a solution to (13). This proves that $\{S_n\}$ constructed from the solution $\{(q_n, r_n)\}$ of the DMNLSE (4) satisfies the discrete equation (11) of the Schrödinger flow of maps into $G_{k,m}$. The proof of the gauge equivalence between the class of solutions to the DCMNLSE (4) and the discrete equation (11) of the Schrödinger flow of maps into $G_{k,m}$ is complete.

Remark 3. It is straightforward to verify that the MNLSE (1) is gauge equivalent to the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (15) by the following gauge transformation:

$$\phi(x,t,\lambda) = G(x,t)\psi(x,t,\lambda) \tag{32}$$

where $G(x,t)$ is a fundamental solution to (8) with $r = q^*$ at $\lambda = 0$, and $\phi(x,t,\lambda)$ and $\psi(x,t,\lambda)$ are solutions to (8) with $r = q^*$ and (14) at λ, respectively. However, we would like to point out that this gauge transformation is somewhat different from the one employed in [2].

Since the continuous limits of Lax pairs (6) and (13) are exactly the classical ones (8) and (14), respectively, and $r = q^*$ from remark 2, it is easy to see that the continuous limit of the gauge transformation (30) between the class of solutions given in remark 1 to the DCMNLSE (4) and the discrete equation (11) of the Schrödinger flow of maps into the

Grassmannian is just a classical gauge transformation (32) between the MNLSE (1) and the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (15).

What remains is to show the conclusion mentioned in remark 2. In fact, noting that the continuous limit of unitary sequence $\{U_n\}$ is again a unitary matrix U, i.e. $U_n \to U$, and combining this with (24), we see that $p_n \to p\Delta x$ for the p with $\begin{pmatrix} 0 & p^* \\ -p & 0 \end{pmatrix} - U_x U^*$ being a diagonal matrix. Here we would like to point out that $U_x U^* \in u(m)$ and is of the form $\begin{pmatrix} S_1 & p^* \\ -p & S_2 \end{pmatrix}$, for some $S_1 \in u(k)$ and $S_2 \in u(m-k)$. It is obvious that the continuous limit G of G_n is $G = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} U$, where the non-singular matrix A (resp. B) is the continuous limit of A_n (resp. B_n). As shown in section 2, we see from (22) and (23) that G satisfies $G_x = \begin{pmatrix} 0 & Ap^*B^{-1} \\ -BpA^{-1} & 0 \end{pmatrix} G$ and hence

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_x = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \left[\begin{pmatrix} 0 & p^* \\ -p & 0 \end{pmatrix} - U_x U^* \right].$$

This implies that A (resp. B) satisfies $A_x = -AS_1$ (resp. $B_x = -BS_2$), where $S_1 \in u(k)$ (resp. $S_2 \in u(m-k)$) as mentioned above. It is easy to see the restriction that the continuous limits of A_0 and B_0 (i.e. $\lim_{\Delta x \to 0} A_0$, etc) are unitary matrices is equivalent to saying that $A|_{x=0}$ and $B|_{x=0}$ are unitary matrices. Therefore A and B have to be unitary under the restriction of ordinary differential equations. This converse statement is clearly true.

4. Example

As an example to illustrate that, in the general case, $r_n \neq q_n^*$ for a couple of the sequences $\{(q_n, r_n)\}$ given in remark 1, we take a trivial solution $\{S_n\}$ to the discrete equation of the Schrödinger flow of maps into the Grassmannian $G_{2,3} = CP^2$ of the form $S_n = \begin{pmatrix} 1 & 0 \\ 0 & s_n \end{pmatrix}$, where $\{s_n\}$ is a 1-soliton solution to the DHF (with the boundary condition $s_n \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $n \to \infty$) which is gauge equivalent to the 1-soliton solution $\psi_n = \exp[-\mathrm{i}(2 - 2\mathrm{ch}w)t] \frac{\mathrm{sh}w}{\mathrm{ch}nw}$ ($w = \Delta x$ being the discretization parameter; see [9]) of the DNLSE$^+$. That is, there is a solution $\{F_n\}$ to

$$\begin{aligned} F_{n+1} &= \begin{pmatrix} 1 & \bar{\psi}_n \\ -\psi_n & 1 \end{pmatrix} F_n \\ \mathrm{d}F_n/\mathrm{d}t &= \mathrm{i} \begin{pmatrix} -\bar{\psi}_n \psi_{n-1} & -\bar{\psi}_n + \bar{\psi}_{n-1} \\ -\psi_n + \psi_{n-1} & \psi_n \bar{\psi}_{n-1} \end{pmatrix} F_n \end{aligned} \tag{33}$$

and $\{s_n\}$ is given by the formula $s_n = \begin{pmatrix} s_n^1 & s_n^2 - \mathrm{i}s_n^3 \\ s_n^2 + \mathrm{i}s_n^3 & -s_n^1 \end{pmatrix} = F_n^{-1}\sigma_3 F_n$. The explicit expression of $\{s_n\}$ will be given below. For this solution $\{S_n\}$, it is easy to verify that, $\forall n$, the following 3×3 matrix $G_n = \begin{pmatrix} 1 & a \\ 0 & F_n \end{pmatrix}$ satisfies

$$G_{n+1} = \begin{pmatrix} I_2 & r_n \\ -q_n & 1 \end{pmatrix} G_n \qquad \mathrm{d}G_n/\mathrm{d}t = \mathrm{i} \begin{pmatrix} -r_n q_{n-1} & -r_n + r_{n-1} \\ -q_n + q_{n-1} & q_n r_{n-1} \end{pmatrix} G_n$$

where $a = (w, 0)F_n$, $r_n = \begin{pmatrix} w\bar{\psi}_n \\ \bar{\psi}_n \end{pmatrix}$ and $q_n = (0, \psi_n)$. Obviously, $\{(q_n, r_n)\}$ is a couple of the sequences in remark 1 for $k = 2$ and $m = 3$ satisfying $r_n \neq q_n^*$, $\forall n$ and $r = q^*$ after taking the continuous limit.

We end this section by giving the explicit expression of the 1-soliton solution $\{s_n\}$ to the DHF mentioned above. In fact, a solution to (33) is $F_n = \begin{pmatrix} f_n & g_n \\ -\bar{g}_n & \bar{f}_n \end{pmatrix}$, where

$$f_n = (c_n^1 \cos(\mathrm{ch}w - 1)t + c_n^2 \sin(\mathrm{ch}w - 1)t) \exp[\mathrm{i}(1 - \mathrm{ch}w)t]$$
$$g_n = (b_n^1 \cos(\mathrm{ch}w - 1)t + b_n^2 \sin(\mathrm{ch}w - 1)t) \exp[\mathrm{i}(1 - \mathrm{ch}w)t]$$

in which $c_n^1 = \mathrm{Re}\,\Pi_{j=0}^{n-1}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)c_0^1 - \mathrm{Im}\,\Pi_{j=0}^{n-1}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)\bar{b}_0^1$, $b_n^1 = \mathrm{Re}\,\Pi_{j=0}^{n-1}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)b_0^1 + \mathrm{Im}\,\Pi_{j=0}^{n-1}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)\bar{c}_0^1$, $c_n^2 = -\mathrm{i}(c_n^1 R_n + \bar{b}_n^1 L_n)/(\mathrm{ch}w - 1)$, $b_n^2 = \mathrm{i}(\bar{c}_n^1 L_n - b_n^1 R_n)/(\mathrm{ch}w - 1)$, and $R_n = 1 - \mathrm{ch}w + \mathrm{sh}^2 w/(\mathrm{ch}nw\,\mathrm{ch}(n-1)w)$, $L_n = -\mathrm{sh}w/\mathrm{ch}nw + \mathrm{sh}w/\mathrm{ch}(n-1)w$. In order to guarantee that s_n satisfies the boundary condition, equivalently, $g_n \to \infty$ as $n \to \infty$, we choose c_0^1 and b_0^1 such that $b_\infty^1 = \lim_{n\to\infty} b_n^1 = 0$. For example, $c_0^1 = \mathrm{Re}\,\Pi_{j=0}^{\infty}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)$ and $b_0^1 = -\mathrm{Im}\,\Pi_{j=0}^{\infty}(1+\mathrm{i}\,\mathrm{sh}w/\mathrm{ch}jw)$. Then the corresponding 1-soliton solution to the DHF is given by $s_n^1 = (|f_n|^2 - |g_n|^2)/(|f_n|^2 + |g_n|^2)$, $s_n^2 = (\bar{g}_n f_n + g_n \bar{f}_n)/(|f_n|^2 + |g_n|^2)$ and $s_n^3 = (\bar{g}_n f_n - g_n \bar{f}_n)/\mathrm{i}(|f_n|^2 + |g_n|^2)$.

5. Conclusion and remarks

In this paper, we have revealed that the class of solutions given in remark 1 to the DCMNLSE (4) (for $k \geqslant 2$ or $m-k \geqslant 2$) is gauge equivalent to the discrete equation (11) of the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ and have demonstrated further that the continuous limit of the realizing gauge transformation is exactly a classical limit between the MNLSE (1) and the Schrödinger flow of maps into the Grassmannian $G_{k,m}$ (15). In other words, from the viewpoint of gauge equivalence, the class of solutions to the DCMNLSE (4) is a correct discretization of the MNLSE (1). However, we have not been able to find a non-trivial example to illustrate that the discrete equation (11) of the Schrödinger flow of maps into the Grassmannian is not, in general, gauge equivalent to the DMNLSE (2), though we believe that, unlike the fact displayed in [19] for the NLSE, this is the case. Anyway, the obtained result suggests that there might exist an interesting and intriguing geometric relationship between the CMNLSE (resp. DCMNLSE) and the MNLSE (resp. DMNLSE). A better understanding of this will be left for future study.

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